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A Reduced System of Differential Equations for the Invariants of Ternary Forms

FLORENCE BLACK
Department of Mathematics, University of Kansas

1. INTRODUCTION. Let C_n represent a general ternary form of order n in the variables x_1, x_2, x_3, and let T denote the general linear transformation

$$T = \begin{cases} x_1 = a_{11}\,\xi_1 + a_{12}\,\xi_2 + a_{13}\,\xi_3, \\ x_2 = a_{21}\,\xi_1 + a_{22}\,\xi_2 + a_{23}\,\xi_3, \\ x_3 = a_{31}\,\xi_1 + a_{32}\,\xi_2 + a_{33}\,\xi_3, \end{cases} \qquad |a_{ik}| \neq 0.$$

It is well known that the necessary and sufficient condition that a homogeneous function F of the coefficients of C_n be invariant under T, is that F be a solution of a certain system of nine linear partial differential equations of the first order, each coefficient of T giving rise to an equation of the system.

By the use of the special transformation

$$\begin{aligned} x_1 &= \xi_1 + a_{12}\,\xi_2 + a_{13}\,\xi_3, \\ x_2 &= \qquad\quad \xi_2 \qquad\qquad , \\ x_3 &= \qquad\qquad\qquad\quad \xi_3, \end{aligned}$$

Junker* found solutions for two of the nine equations and by the introduction of these solutions as new variables reduced the number of equations in the complete system to seven.

It is the purpose of this work to reduce the number of equations in the system to six. We use the special transformation

$$(1) \quad T_1 = \begin{cases} x_1 = \xi_1 + a_{12}\,\xi_2 + a_{13}\,\xi_3, \\ x_2 = \qquad\quad \xi_2 + a_{23}\,\xi_3, \\ x_3 = \qquad\qquad\qquad\quad \xi_3, \end{cases}$$

and define a seminvariant of the ternary form as any homogeneous function of the coefficients which is invariant under T_1.

* Math. Annalen, vol. 64, pp. 328-343.

2. CALCULATION OF THE SEMINVARIANTS. The general ternary *n-ic*, C_n may be written as a determinant* with linear elements in the form

$$(2)\quad C_n = \begin{vmatrix} x_1 + a_{11}x_2 + b_{11}x_3 & a_{12}x_2 + b_{12}x_3 & \cdots \\ a_{21}x_2 + b_{21}x_3 & x_1 + a_{22}x_2 + b_{22}x_3 & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1}x_2 + b_{n1}x_3 & & x_1 + a_{nn}x_2 + b_{nn}x_3 \end{vmatrix}.$$

When expanded C_n may be written

$$(3)\quad I^{no}x_1^n + I^{n-1,0}x_1^{n-1}x_2 + I^{n-1,1}x_1^{n-1}x_3 + \cdots + I^{ik}x_1^i x_2^j x_3^k,$$

where $I^{no} = 1$ and $I^{ro}(r<n)$ is the sum of all principal minors of order $n - r$ of the determinant

$$A = \begin{vmatrix} a_{11} & \cdots \\ \cdots & \cdots \\ \cdots & a_{nn} \end{vmatrix},$$

and I^{rs} $(r + s \leqq n)$ is the sum of all possible determinants obtained by replacing s columns at a time in each of the principal minors of I^{ro} by the same columns from the corresponding minors of

$$B = \begin{vmatrix} b_{11} & \cdots \\ \cdots & \cdots \\ \cdots & b_{nn} \end{vmatrix}.$$

For example $I^{00} = A$, and I^{01} is the sum of the n determinants of the form

$$\begin{vmatrix} b_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ b_{21} & a_{22} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & a_{n2} & \cdots & & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & b_{22} & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & b_{n2} & \cdots & & a_{nn} \end{vmatrix} + \cdots.$$

The coefficients I^{rs} may be obtained from $A = I^{00}$ by means of two differential operators†

$$D_a = \sum_{i=1}^{n} \frac{\partial}{\partial a_{ii}} \text{ and } D_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \frac{\partial}{\partial a_{ij}}.$$

* Dickson, Transactions of the American Math. Soc., vol. 22, pp. 167-179; Stouffer, *ibid.*, vol. 26, No. 3, pp. 356-368.

† cf., Stouffer, loc. cit., p. 363.

In general for the ternary form

$$D_a I^{rs} = (r+1)\, I^{r+1,\,s}\,,$$

and

$$D_{ab} I^{rs} = (s+1)\, I^{r,\,s+1}\,.$$

If we represent the result of k successive applications of D_a upon I^{rs} by $D_a^k I^{rs}$ and of D_{ab} on I^{rs} by $D_{ab}^k I^{rs}$ we have

$$D_a^k I^{rs} = \frac{(r+k)\,!}{r\,!}\, I^{r+k,\,s}\,,$$

and

$$D_{ab}^k I^{rs} = \frac{(s+k)\,!}{s\,!}\, I^{r,\,s+k}\,.$$

In terms of I^{rs} the differential equations* satisfied by the invariants of a ternary *n-ic* may be written in the following form. The summations extend over the coefficients of C_n.

$$E_1 = \Sigma\Sigma\,(i+1)\, I^{i+1,k} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_2 = \Sigma\Sigma\,(i+1)\, I^{i+1,k-1} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_3 = \Sigma\Sigma\,(n-i+1-k)\, I^{i-1,k} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_4 = \Sigma\Sigma\,(k+1)\, I^{i-1,k+1} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$(4)\quad E_5 = \Sigma\Sigma\,(n-i-k+1)\, I^{i,k-1} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_6 = \Sigma\Sigma\,(k+1)\, I^{i,k+1} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_7 = \Sigma\Sigma\,(i-1)\, I^{ik} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_8 = \Sigma\Sigma\,(n-1-2k)\, I^{ik} \frac{\partial f}{\partial I^{ik}} = 0,$$

$$E_9 = \Sigma\Sigma\,(n-2i-k)\, I^{ik} \frac{\partial f}{\partial I^{ik}} = 0.$$

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* E. B. Elliott, Algebra of Quantics, p. 379.

After transformation (1) has been applied to C_n in form (2) the finite transformations of the elements a and b may be built, and from these their infinitesimal transformations calculated. The resulting infinitesimal transformations of a and b

$$\begin{aligned}
\delta a_{ii} &= \Phi_{12}\,\delta t, \\
\delta a_{ij} &= 0 \quad (i \neq j), \\
\delta b_{ii} &= (\Phi_{13} + \Phi_{23} a_{ii})\,\delta t, \\
\delta b_{ij} &= (\Phi_{23} a_{ij})\,\delta t \quad (i \neq j),
\end{aligned}$$

are found to have the same form as the infinitesimal transformations obtained by Stouffer* when working on an entirely different problem, that of seminvariants and invariants of a system of linear homogeneous differential equations.

The differential equations considered by Stouffer are of the form

$$y''_i + \sum_{k=1}^{n} (2p_{ik}\, y'_k + q_{ik}\, y_k) = 0 \quad (i = 1, 2, \cdots, n),$$

where the primes indicate differentiation with respect to the independent variable x, and p_{ik} and q_{ik} are functions of x. A function of the coefficients and their derivatives which is unchanged by the general transformation of the dependent variables

$$(5) \quad y_k = \sum_{\lambda=1}^{n} a_{k\lambda}(x)\, \eta_\lambda \quad (k = 1, 2, \cdots, n),$$

is defined as a seminvariant, and if this function is also invariant under the general transformation of the independent variable $\xi = \xi(x)$, it is called an invariant of the system. The seminvariants of the system or the invariants under the group formed by transformations (5), are solutions of the system of partial differential equations obtained by equating to zero the most general infinitesimal transformation of the group.

Determinants having the same form as our coefficients in (3) are shown to be seminvariants, the infinitesimal transformations of which are identical in form with the infinitesimal transformations of our coefficients I^{rs}. With certain minor changes in constants the seminvariants of our problem are the same functions of the coefficients I^{rs} as the invariants of Stouffer's problem are of his seminvariants. Consequently, by the use of Stouffer's results we know that the following functions are seminvariants of our ternary $n - ic$, facts which may also be verified directly by means of equations (4).

* Stouffer, Proceedings of the London Math. Soc., Sec. 2, vol. 15, Part 3, and Sec. 2, vol. 17, Part 5.

$$(6)\quad A_\alpha = (I^{n-1,0})^\alpha - \alpha! \sum_{i=2}^{\alpha-1} \frac{1}{(\alpha-i)!\,i!} (I^{n-1,0})^{\alpha-i} A_i$$

$$- \frac{\alpha!\, n^{\alpha-1}}{(n-1)(n-2)\ldots(n-\alpha+1)} I^{n-\alpha,0} (I^{n0})^{\alpha-1} \quad (\alpha = 2, 3, \ldots, n),$$

$$(7)\quad A_{\alpha\beta} = 2A_2 D_{ab} A_{\alpha,\beta-1} - \alpha A_{\alpha,\beta-1} D_{ab} A_2$$

$$(\alpha = 3, 4, \cdots, n; \beta = 1, 2, \cdots, \alpha),$$

$$(8)\quad B = 4A_2 (I^{n-1,1})^2 - \frac{8n}{n-1} I^{n-2,2} I^{n0} A_2 - (D_{ab} A_2)^2.$$

The seminvariants A_α, $A_{\alpha\beta}$, and B are independent, for if they are arranged in the order A_α, B, and $A_{\alpha\beta}$, and according to ascending values of α, and β, each contains a coefficient I^{rs} not in those which precede.

3. THE DIFFERENTIAL EQUATIONS IN TERMS OF THE SEMINVARIANTS. The expressions A_α, $A_{\alpha\beta}$, B, and I^{no}, $A_2 \neq 0$, may be introduced as new variables into the set of nine differential equations (4), and the new coefficients expressed in terms of these seminvariants. Equations E_1, E_2, and E_5 are satisfied identically by the new variables and thus the number of equations in the set will be reduced by three. In order to introduce* the new variables I^{no}, A_α, $A_{\alpha\beta}$, and B, into any other of the partial differential equations $E_i = 0$, we operate with E_1 on A_α, $A_{\alpha\beta}$, B, and I^{no}. Equation $E_i = 0$, in terms of the new variables, will read

$$\sum E_i(A_\alpha) \frac{\partial f}{\partial A_\alpha} + E_i(A_{\alpha\beta}) \frac{\partial f}{\partial A_{\alpha\beta}} + E_i(B) \frac{\partial f}{\partial B}$$

$$+ E_i(I^{no}) \frac{\partial f}{\partial I^{no}} = 0,$$

where by $E_i(F)$ we represent the results obtained by operating on F by E_i.

After having operated with E_i upon the seminvariants, we may as a matter of simplification set $I^{n-1,0} = I^{n-1,1} = I^{n-2,1} = 0$. This may be done without loss of generality since by choosing in transformation (1)

$$a_{12} = -\frac{I^{n-1,0}}{nI^{n0}},$$

$$a_{13} = \frac{2I^{n-1,1} I^{n-2,0} - I^{n-2,1} I^{n-1,0}}{(n-1) A_2},$$

$$a_{23} = \frac{nI^{n-2,1} I^{n0} - (n-1) I^{n-1,0} I^{n-1,1}}{(n-1) A_2},$$

* Horn, Einfuhrung in die Theorie der partiellen differential-gleichungen.

the three transformed coefficients $\overline{I^{n-1,0}}$, $\overline{I^{n-1,1}}$ and $\overline{I^{n-2,1}}$ of the resulting *n-ic,* will be reduced to zero without having changed the value of the seminvariants.

We shall consider in detail only

$$E_3 = \Sigma\Sigma\,(n-i+1-k)\,I^{i-1,k}\frac{\partial f}{\partial I^{ik}} = 0.$$

If we operate upon the seminvariants with E_3 and simplify, we have:

$$E_3\,(I^{n0}) = 0,$$

$$E_3\,(A_a) = \frac{a(n-1)}{n}A_2\,A_{a-1} + \frac{(n-a)}{n}A_{a+1}\,,$$

$$E_3\,(B) = \frac{(n-2)\,A_{32}}{6n\,A_2} + \frac{3(n-2)\,A_3\,B}{2n\,A_2}\,,$$

$$(9)\quad E_3\,(A_{a\beta}) = \frac{(n-2)\,A_3\,A_{a\beta}}{n\,A_2} - \frac{a\,(n-2)\,A_{a,\beta-1}\,A_{31}}{3n\,A_2} + 2A_2\,E_3\,(D_{ab}\,A_{a,\beta-1}).$$

If $\beta = 1$ in (9) the last term may also be expressed directly in terms of the new variables and we have

$$E_3\,(A_{a1}) = \frac{(n-2)\,A_3\,A_{a1}}{n\,A_2} + \frac{a(n-1)\,A_2\,A_{a-1,1}}{n} + \frac{a(n-a)\,A_{a+1,1}}{n(a+1)} - \frac{a(n-2)\,A_a\,A_{31}}{3n\,A_2}.$$

If $\beta = 2$ we have

$$E_3\,(A_{a2}) = \frac{(2-2a)\,(n-2)\,A_{31}\,A_{a1}}{3n\,A_2} + \frac{2(n-2)\,A_3\,A_{a2}}{n\,A_2} + \frac{a(n-2)\,A_3\,A_a\,B}{2n\,A_2} + \frac{(a-1)\,(n-a)\,A_{a+1,2}}{n(a+1)} - \frac{(n-a)\,A_{a+1}\,B}{n} + \frac{a(n-1)\,A_{a-1,2}\,A_2}{n} - \frac{a(n-1)\,A_{a-1}\,A_2\,B}{n} - \frac{a(n-2)\,A_{32}\,A_a}{6n\,A_2}.$$

In the general case it is necessary to prove that $E_3\,(D_{ab}\,A_{a,\beta-1})$ may for any particular value of β be expressed in terms of the new variables. The proof of this fact may be given in three parts.

(I) We shall first show that $D_{ab}\,(A_{a\beta})$ may be expressed as a function of A_a, $D_{ab}\,A_a, \cdots D^q_{ab}\,A_a$. If we can find a value of γ

such that $A_{a\gamma}$ is a function of (10) A_a, $D_{ab} A_a \cdots D^r_{ab} A_a$, A_2, $D_{ab} A_2$, $D^2_{ab} A_2$, $(a>2)$ then from (7) it follows that $A_{a,\gamma+1}$ is a function of (10) and $D^{r+1}_{ab} A_a$. But we may write A_{a2} in the form

$$A_{a2} = 2A_2 D_{ab}(2A_2 D_{ab} A_a - aA_a D_{ab} A_2) - a(2A_2 D_{ab} A_a - aA_a D_{ab} A_2) D_{ab} A_2.$$

It now follows that $A_{a3}, A_{a4}, \cdots$ and finally $A_{a\beta}$ may be expressed as a function of (10). Thus (I) follows immediately.

(II) We shall next show that

$$E_3 [D^q_{ab}(A_a)] = \frac{(a+1-q)(n-a)}{n(a+1)} D^q_{ab} A_{a+1} - \frac{a(n-1)}{n} A_2 D^q_{ab} A_{a-1} \quad (q=1,2\cdots,a).$$

By direct operation with D_{ab} upon (6) we obtain

$$D_{ab}(A_a) = a(I^{n-1,0})^{a-1} I^{n-1,1} - a!\sum_{i=2}^{a-2} \frac{(a-i)}{(a-i)!\,i!} (I^{n-1,0})^{a-i-1} I^{n-1,1} A_i$$

$$- a!\sum_{i=2}^{a-2} \frac{(I^{n-1,0})^{a-i}}{(a-i)!\,i!} D_{ab}(A_i) - a I^{n-1,1} A_{a-1}$$

$$- a I^{n-1,0} D_{ab} A_{a-1} - \frac{a!\,n^{a-1} I^{n-a,1} (I^{n0})^{a-1}}{(n-1)(n-2)\ldots(n-a+1)}.$$

A second operation gives us $D^2_{ab}(A_a) = -2a I^{n-1,1} D_{ab} A_{a-1}$

$$- a I^{n-1,0} D^2_{ab} A_{a-1} - \frac{2a!\,n^{a-1} I^{n-a,2} (I^{n0})^{a-1}}{(n-1)(n-2)\ldots(n-a+1)}$$

$+$ (... a series of terms of second or higher degree in $I^{n-1,0}$ and $I^{n-1,1}$).

By induction we may show that $D^q_{ab}(A_a) = -qa I^{n-1,1} D^{q-1}_{ab} A_{a-1}$

$$- a I^{n-1,0} D^q_{ab} A_{a-1} - \frac{q!\,a!\,n^{a-1} I^{n-a,q} (I^{n0})^{a-1}}{(n-1)(n-2)\ldots(n-a+1)}$$

$+$ (... terms of second or higher degree in $I^{n-1,0}$ and $I^{n-1,1}$). If we operate on the above by E_3 and set $I^{n-1,0} = I^{n-1,1} = I^{n-2,1} = 0$, we have (II).

(III.) We shall show finally that $D^q_{ab}(A_a)$ $(q=1,2,\ldots,a)$, may

be expressed as a function of the seminvariants A_α, $A_{\alpha\beta}$, and B. If we apply the operator D_{ab} to A_α and $A_{\alpha\beta}$ and set $I^{n-1,0} = I^{n-1,1} = I^{n-2,1} = 0$, we have

$$D_{ab}(A_2) = 0,$$

$$D^2_{ab}(A_2) = \frac{B}{2A_2},$$

$$D_{ab}(A_2) = 0,$$

$$D_{ab}(A_\alpha) = \frac{A_{\alpha 1}}{2A_2} \quad (\alpha = 3, \ldots, n),$$

$$(11)\quad D^2_{ab}(A_\alpha) = \frac{1}{4A_2^2}\,[A_{\alpha 2} + 2(\alpha - 1)\,A_{\alpha 1}\,D_{ab}\,A_2$$

$$+ \alpha(\alpha - 2)\,A_\alpha\,(D_{ab}\,A_2)^2 + 2\alpha\,A_2\,A_\alpha\,D^2_{ab}\,A_2],$$

$$D_{ab}(A_{\alpha\beta}) = \frac{A_{\alpha,\beta+1}}{2A_2} \quad [\beta = 1, \ldots, (\alpha - 1)].$$

If we can find a value for r such that $D^r_{ab}(A_\alpha)$ may be expressed as a function of

(12) A_2, A_α, $A_{\alpha 1}$, $A_{\alpha 2}$ ... $A_{\alpha r}$, $D_{ab}\,A_2$, $D_{ab}\,A_2$, then by means of the above results $D^{r+1}_{ab}(A_\alpha)$ $(r = 1, 2, \ldots, \alpha - 1)$, may in turn be expressed as a function of A_2, A_α, $A_{\alpha\beta}$, and B. In (11) we have $D^2_{ab}(A_\alpha)$ defined in terms of functions (12). Thus by induction (III) follows.

By (I), (II), and (III), it now follows that $E_3\,(D_{ab}\,A_{\alpha,\beta-1})$ may be expressed in terms of the seminvariants, thus completing the proof that $E_3\,(A_{\alpha\beta})$ may be expressed in terms of the seminvariants as new variables.

The method of introduction of A_α, $A_{\alpha\beta}$, B, and I^{no} into the remaining differential equations of set (4) parallels very closely that used for E_3.

As a result we have the following six partial differential equations which for $A_2 \neq 0$ must be satisfied by the invariants of a ternary *n-ic*. The summations extend over all subscripts α and β.

$$E_3 = \Sigma\left[\frac{\alpha(n-1)}{n}\,A_2\,A_{\alpha-1} + \frac{n-\alpha}{n}\,A_{\alpha+1}\right]\frac{\partial f}{\partial A_\alpha}$$

$$+\left[\frac{(n-2)\,A_{32}}{6n\,A_2} + \frac{3(n-2)\,A_3\,B}{A_2}\right]\frac{\partial f}{\partial B}$$

$$+\Sigma\Sigma\left[\frac{(n-2)\,A_3\,A\alpha_{-\beta}}{n\,A_2}-\frac{\alpha(n-2)\,A\alpha_{\beta-1}\,A_{31}}{3n\,A_2}\right.$$

$$\left.+2A_2\,E_3\,(D_{\mathrm{ab}}\,A\alpha_{,\beta-1})\right]\frac{\partial f}{\partial A\alpha_\beta}=0,$$

$$E_4+\Sigma\left[\frac{(n-\alpha)\,A_{\alpha+1,1}}{2n\,(\alpha+1)\,A_2}\right]\frac{\partial f}{\partial A\alpha}+\left[\frac{(n-2)\,A_{33}}{12n\,A_2^2}\right.$$

$$\left.+\frac{3(n-2)\,A_{31}\,B}{4n\,A_2^2}\right]\frac{\partial f}{\partial B}$$

$$+\Sigma\Sigma\left[\frac{(n-2)\,A_{31}\,A\alpha_\beta}{6n\,A_2^2}-\frac{\alpha(n-2)\,A_{32}\,A\alpha_{,\beta-1}}{12n\,A_2^2}\right.$$

$$\left.-\frac{\alpha(n-2)\,A_3\,B\,A\alpha_{,\beta-1}}{4n\,A_2^2}+2A_2\,E_4\,(D_{\mathrm{ab}}\,A\alpha_{,\beta-1})\right]\frac{\partial f}{\partial A\alpha_\beta}=0.$$

$$E_6=\Sigma\left(\frac{A_{\alpha 1}}{2A_2}\right)\frac{\partial f}{\partial A_\alpha}+\Sigma\Sigma\,D_{\mathrm{ab}}\,(A\alpha_\beta)\,\frac{\partial f}{\partial A_{\alpha\beta}}=0,$$

$$E_7=n\,I^{\mathrm{n0}}\frac{\partial \mathrm{f}}{\partial I^{\mathrm{no}}}+\Sigma\,\alpha(n-1)\,A\alpha\frac{\partial \mathrm{f}}{\partial A\alpha}+(4n-6)\,B\frac{\partial \mathrm{f}}{\partial B}$$

$$+\Sigma\Sigma\,[2(n-1)\,A\alpha_\beta+2A_2\,E_7\,(D_{\mathrm{ab}}\,A\alpha_{,\beta-1})]\,\frac{\partial f}{\partial A\alpha_\beta}=0,$$

$$E_8=\Sigma\alpha\,A\alpha\frac{\partial f}{\partial A\alpha}+\Sigma\Sigma\,[2A\alpha_\beta$$

$$+2A_2\,E_8\,(D_{\mathrm{ab}}\,A_{\alpha,\beta-1})]\frac{\partial f}{\partial A\alpha_\beta}=0,$$

$$E_9=\Sigma\,(2\alpha-\alpha n)\,A_2\frac{\partial f}{\partial A\alpha}-n\,I^{\mathrm{n0}}\frac{\partial f}{\partial I^{\mathrm{no}}}+(6-4n)\,B\frac{\partial f}{\partial B}$$

$$+\Sigma\Sigma\,[(4-2n)\,A\alpha_\beta+2A_2\,E_9\,(D_{\mathrm{ab}}\,A_{\alpha,\beta-1})]\,\frac{\partial f}{\partial A\alpha_\beta}=0.$$